maps the open rectangle onto the prescribed cone. With $z(x,y)$ and $r(x,y)$ defined by Eq. (5), the finite difference equations corresponding to Eqs. (3) and (4) were solved on a rectangle using the method previously described.

For short horns with slowly varying cross sections, the catenoidal and exponential horns of Morse[8] are approximately the same as a hyperbolic horn and a horn of the shape $y = be^{\pi x/2D} + D$. The agreement with Morse for these cases was extremely good; however, the restriction to slowly varying cross sections lead to solutions differing little from plane wave solutions. As an indication of the close agreement with Morse, the respective attenuation at $x=1$ for the expotential horns were -1.9db for Morse's solution and -1.7db for the finite difference solution. It must be emphasized that the restriction to horns with slowly varying cross sections was for the purpose of approximating Morse's horn shapes and not because of limitations of the finite difference method

Results

For a conical horn with a hard wall, the exact solution is known. For $\alpha = 0.75$, $R = 0.5$, $\eta = 1$, and $\rho^2 + r^2$ the solution is

$$p(z,r) = -0.5(\cos 2\pi\rho - i \sin 2\pi\rho)/\rho \qquad (6)$$

In Figs. 2 and 3, the real and imaginary parts of the finite difference solution are compared to those of the exact solution (6). A study was made for the same conical horn with a uniform lining. The results are summarized in Fig. 4 where one readily sees that the best uniform lining is one with a resistance of 0.2 and a reactance of -0.5.

Conclusion

The flexibility of the finite difference approach yields results for ducts of nonuniform cross section in the no flow case. For a hard wall, the agreement with the exact solution is outstanding. The finite difference method can also be used to compute attenuations due to nonuniform linings in both uniform and variable area ducts, as well as uniform flow in a variable area duct.

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Calculating Starting Times for a Supersonic Nozzle Upstream of a Diaphragm

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Nomenclature

a = speed of sound
A = cross-sectional area
k = ratio of specific heats
p = pressure
t = $t_1 + t_2 + t_3$
w = flow velocity
z = distance from diaphragm to station z in nozzle or test section
ρ = density

Subscripts

c = wave point c
e = wave point e at station z
D = diaphragm location
n = test section
o = stagnation
t = upstream of nozzle
x = upstream of stationary normal shock at station z
y = downstream of stationary normal shock at station z
z = downstream of moving normal shock at station z
4 = upstream of diaphragm before rupture
* = nozzle throat

Introduction

In all short duration test facilities, knowledge of the starting times is important in determining quasi-steady flow requirements. It is generally recognized[1] that a diaphragm downstream of the nozzle causes a longer starting time than an upstream diaphragm configuration, and hence adequate prediction of the starting times is essential for this case. A simple, experimentally verified procedure for estimating the starting times of a nozzle with a downstream diaphragm is presented herewith. The method is based upon the results (using dry air) from the Mach 1.4, 2, 3, and 4 two-dimensional nozzles of the Leeds University Supersonic Ludwieg Tube.[2] The estimates are seen to become more accurate as the nozzle Mach number increases.

The starting process (approximated in the x-t diagram of Fig. 1) commences with the expansion wave, formed at diaphragm burst, traveling upstream through the test section and into the convergent-divergent nozzle. This wave becomes intensified with decreasing cross section (as seen from the downstream diaphragm end) and, after the throat, becomes attenuated again in the divergent part. As this rarefaction wave travels upstream it accelerates the gas in a downstream direction. Sonic velocity is first reached at the nozzle throat where the wave possesses its greatest intensity. At this instant no further part of the wave can pass upstream through the nozzle throat (since expansion waves travel at local sonic velocity). As soon as the nozzle throat becomes choked a weak

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shock forms at this point and the wave processess become more complicated. As more and more of the expansion wave, centered at the diaphragm location, reaches the starting shock, the shock moves to the nozzle exit. The remainder of the expansion wave thereupon weakens the shock and it is eventually swept out of the test section. It is convenient, therefore, to divide the starting time t into three phases: the time required for nozzle choking t_1; the time for the starting shock formed at the nozzle throat to reach the end of the nozzle t_2; the time for the shock to travel the length of the test section t_3.

The method of one-dimensional characteristics[3] had previously been used to predict values of the starting times of the MSFC high Reynolds number wind tunnel.[4] The results gave reasonable agreement with the test data except at the higher supersonic Mach numbers. This difference at the higher Mach numbers was attributed to the two- and three-dimensional character of the flow. Because of these multi dimensional effects, it is apparent that the results predicted by a simple one-dimensional theory would, in all probability, be no less accurate than those obtained by any other more complex one-dimensional theory. One such simple method has been used by Falk[1] for zero length nozzles. In his analysis an assumed normal starting shock remains stationary in the nozzle before entering the test section with zero velocity. In the present investigation the simple one-dimensional finite amplitude wave theory is used in a more realistic approach. This method incorporates a semiempirical formula, which allows the flow conditions downstream of a stationary normal starting shock to be perturbed, permitting the shock to travel downstream at a known velocity.

Estimation of Starting Times

To maintain simplicity in estimating the starting times it is necessary to make the general assumption that the expansion wave remains unaltered by any reflection processes due to the nozzle area change. More specific assumptions are needed for each of the three parts into which the starting process is divided.

An estimate of the time t_1 (Fig. 1) can be determined by finding the time at which that part of the expansion wave, transmitted through the nozzle, accelerates the flow immediately upstream of the nozzle to the Mach number M_t required for choking. (M_t is determined by the geometry of the test facility.) The flow velocity at wave point c (corresponding to the choking characteristic of Fig. 1) where the flow Mach numbers is M_t, is given by

$$w_c = [-2a_4/(k-1)][(1+M_t(k-1)/2)^{-1}-1] \tag{1}$$

Wave point c travels to the nozzle throat with velocity

$$V_1 = a_c - w_c \tag{2}$$

Then

$$t_1 = L_1/V_1 \tag{3}$$

where L_1 is the distance from the diaphragm to the nozzle throat (neglecting the convergent part of the nozzle).

The subsonic conditions just downstream of the starting shock, at any position in the nozzle, are not easily defined. These conditions are formed by a combination of a deceleration through the shock, an acceleration due to the remainder of the expansion wave not transmitted into the storage tube, and an effect due to the reflection of the expansion inside the nozzle. To calculate t_2 (Fig. 1) the starting shock must be assumed normal and an approximation must be made to the downstream flowfield. The method is to consider a stationary shock at any point in the nozzle, and then to include an effect from the expansion wave, perturbing the downstream conditions, such that the shock moves to the nozzle exit.

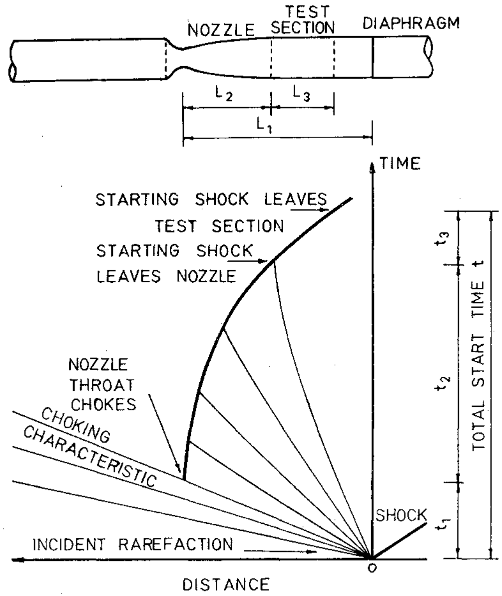

Fig. 1 Approximate wave diagram of the starting process of a nozzle upstream of a diaphragm.

At any station z in the nozzle the area ratio is known, and hence the Mach number M_x just upstream of a stationary shock is also known. The stagnation conditions upstream of the shock are simply the stagnation conditions upstream of the nozzle (a function of M_t), and so the normal shock relations can be used to find the Mach number M_y and the stagnation pressure p_{0y} downstream of the shock. Hence the static pressure p_y and density ρ_y downstream of the shock can be found. The flow velocity downstream of the stationary shock is thus given by

$$w_y = (kp_y/\rho_y)^{1/2} M_y \tag{4}$$

Provided the pressure in the vacuum tank is low enough, the flow at the diaphragm location due to the expansion wave will be sonic throughout the starting process. Hence

$$w_D = a_D = 2a_4/(k+1) \tag{5}$$

If the pressure ratio $(p_e/p_4)^{(k-1)/2k}$ of the remainder of the expansion wave is assumed linear over the distance between the nozzle throat to the diaphragm location, then the velocity w_e can be assumed to vary linearly with distance also. However, since an expansion wave is an unsteady process, it is convenient to fix the remainder of the expansion wave with respect to time. Linear interpolation gives for this "fixed" expansion wave

$$w_e = w_D + (w_t - w_D)z/L_1 \tag{6}$$

The problem of combining the effect of the expansion wave with the flow downstream of the stationary shock appears to be largely intuitive. A semiempirical combination of the velocities is given following

$$w_z = w_y + (EF)w_e \tag{7}$$

where

$$E = (A_z - A_*)/(A_n - A_*) \quad F = (A_*/A_n)^{1/2}$$

E is simply an area coefficient which varies between 0 and 1 between the nozzle throat and nozzle exit, and thus allows for an increasing contribution from the expansion wave as A_z increases. F represents a measure of the change in kinetic energy across the nozzle and thus takes into account the acceleration in the nozzle.

Since the downstream conditions have been altered, the shock will travel downstream with increasing velocity. Thus, an average shock velocity must be found from the velocities calculated at various distances along the nozzle. If the velocity of the starting shock at station z is V, then mass continuity across the shock gives

$$V=(\rho_* a_* A_* / A_z - \rho_z w_z)/(\rho_x - \rho_z) \quad (8)$$

Neglecting any change in stagnation conditions caused by the flow combination, the Mach number M_z downstream of the moving shock can be found from

$$w_z = (k p_{oy}/\rho_{oy})^{1/2}(1+M_z^2(k-1)/2)^{-1/2} M_z \quad (9)$$

and hence ρ_z can be found.

Mass continuity between the reservoir and the nozzle throat gives

$$\rho_* a_* A_* = w_t A_t (p_t)^{1/k} \quad (10)$$

and hence V can be found. If the average value of the velocity in traveling to the nozzle exit is V_2 then

$$t_2 = L_2 / V_2 \quad (11)$$

where L_2 is the length of the nozzle from throat to exit.

For the shock traveling through the test section, the conditions upstream of it are the given constant test section conditions, while the downstream conditions vary. Using an approach similar to the above, we find the shock velocity V at any point. In this case the area coefficient E in Eq. (7) is always unity. Mass continuity across the shock gives

$$V=(\rho_n w_n - \rho_z w_z)/(\rho_n - \rho_z) \quad (12)$$

Since w_n can be found from the given test section Mach number M_n then V can be calculated. If V_3 is the average shock velocity in the test section then

$$t_3 = L_3 / V_3 \quad (13)$$

where L_3 is the length of the test section.

Results and Comparison with Experiment

Schlieren observations were taken of the starting processes of the 4 nozzles of the Leeds University Supersonic Ludwieg Tube. The measured times of each starting phase are compared with the estimated times in Table 1. (There is no measured value of t_3 for the $M_n=4$ nozzle, since complete starting did not occur.) Furthermore, the previous theory has been applied to the MSFC high Reynolds number axisymmetric tunnel[5] with nitrogen at Mach 1.7 and 3.5

At Mach 2 and above the prediction of the starting times is accurate, even though the unsteady starting flow had a pronounced two-dimensional nature[2] while exhibiting such real gas effects as separation. In the transonic region agreement is poor, and it may be advantageous to revert to the method of characteristics[3] in this region, since the two-dimensional effects are not severe. Improved accuracy is unlikely with Falk's method[1] because the prediction of t_2 using the zero length nozzle analysis is very sensitive to the choice of conditions in the variable entropy region, downstream of the starting shock (e.g., for Falk's $M=3$ nozzle, a 1% change in the nondimensional sound speed at the edge of the variable entropy region causes approximately a 20% change in t_2). This virtually precludes the use of Falk's method, especially at the higher Mach numbers where the nozzle starting time t_2 is dominant.

The errors involved in the estimations of t are difficult to assess. The theoretical model is not truly representative of the physical situation, but the method does have the advantages that the complex flow regimes that actually occur during starting are not relevant to this analysis. Care must be taken in the numerical solution of the shock velocity very near the throat. This is because the equations are ill-conditioned and round-off errors become important near this region.

Table 1 Measured (estimated) starting times in msec

M_n	t_1	t_2	t_3	t
1.4	4.3(3.1)	3.9(1.0)	4.0(1.0)	12.2(5.1)
2.0	3.1(2.6)	6.0(4.8)	4.4(1.9)	13.4(9.3)
3.0	2.9(2.3)	9.1(9.1)	3.0(2.2)	15.0(13.6)
4.0	2.9(2.2)	11.7(14.5)	(2.9)	(19.6)
1.7	(2.1)	(3.2)	(3.0)	21.0(8.3)
3.5	(2.6)	(20.6)	(4.4)	25.0(27.6)

Conclusions

It had been shown previously that a more complicated analysis did not produce a reliable estimate of the nozzle starting times at the higher supersonic Mach numbers. The simpler one-dimensional method of this investigation incorporating a semiempirical equation is useful for predicting the starting times with reasonable accuracy, especially at the higher Mach numbers. These estimates are, in general, only to be used as a fairly rough guide to the expected starting times.

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Impulse Loading of Finite Cylindrical Shells

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SIMULATION of impulse loads produced by radiation induced material blowoff has recently received the attention of several investigators. The loading of rings with a cosine distributed impulse over half the ring circumference has been accomplished with explosives,[1,2] magnetically driven flyer plates,[3] and magnetic pressure pulses.[4] This Note describes a method of loading finite length cylindrical shells

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